Journal of Applied Mechanics and Technical Physics, Vol. 36, No. 2, 1995

METHOD OF LOCAL APPROXIMATIONS IN THE NONLINEAR THEORY OF SHELLS

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UDC 539.3

The method of local approximations, whose basic concepts are given in [1, 2], is based on the representation of the strains of shells in small regions (elements) by Taylor series expansions. If we confine ourselves to a small element and refer its geometry to a certain simple geometrical object, say a plane, the strain relations can be simplified. Depending on the degree of local approximation, various nonlinear models of the strains of shells are obtained. Here we investigate the most effective finite-element models based on the first terms of local approximations.

1. In the neighborhood of a point O the centroidal surface of the shell is defined by the equation

$$\mathbf{r} = \mathbf{r}^p + \mathbf{n}^p\zeta, \quad \mathbf{r}^p = \mathbf{r}^0 + \mathbf{r}_{,i}\xi_i \qquad (i = 1,2), \tag{1.1}$$

where $\mathbf{r}$ is the radius vector of the surface, $\mathbf{r}^p$ is the radius vector of the plane tangent to the shell surface at the point O, $\zeta = \zeta(\xi_1, \xi_2)$ is a function describing shape of the surface in the neighborhood of the point under consideration, ξ_1 and ξ_2 are orthogonal coordinates on the tangent plane, and a subscript after a comma denotes differentiation with respect to the corresponding coordinate. Summation over repeated indexes is used everywhere.

Neglecting small terms $\zeta_{,i}^2 << 1$, we obtain an expression for the normal vector $\mathbf{n}$ to the shell surface

$$\mathbf{n} = (\mathbf{n}^p - \mathbf{r}_{,i}^p\zeta_{,i})(1 + \zeta_{,i}^2)^{-1/2} \approx \mathbf{n}^p - \mathbf{r}_{,i}^p\zeta_{,i}. \tag{1.2}$$

According to the Kirchhoff–Love hypothesis on the preservation of a normal element, we write the components of the strain tensor in the form

$$\begin{gathered}\varepsilon_{ij}^z = \varepsilon_{ij} + z\text{æ}_{ij} = \frac{1}{2}(\mathbf{r}_{,i}^{z\vee}\mathbf{r}_{,j}^{z\vee} - \mathbf{r}_{,i}^z\mathbf{r}_{,j}^z),\\ \mathbf{r}^z = \mathbf{r} + z\mathbf{n}, \qquad \mathbf{r}^{z\vee} = \mathbf{r}^\vee + z\mathbf{n}^\vee.\end{gathered} \tag{1.3}$$

Here $\mathbf{r}^\vee$ and $\mathbf{n}^\vee$ are vectors referring to the deformed state, and z is the normal coordinate to the centroidal surface of the shell.

On the basis of relations (1.1)-(1.3) we define the components ε_{ij} and æ_{ij} for the large displacements and rotations of the centroidal surface of the shell by the local approximate relations

$$\begin{gathered}\varepsilon_{ij} = e_{ij} + \frac{1}{2}(\zeta_{,i}^\vee\zeta_{,j}^\vee - \zeta_{,i}\zeta_{,j}), \quad e_{ij} = \frac{1}{2}(\mathbf{r}_{,i}^{p\vee}\mathbf{r}_{,j}^{p\vee} - \delta_{ij}),\\ \text{æ}_{ij} = -(\zeta^\vee - \zeta)_{,ij} = -w_{,ij},\end{gathered} \tag{1.4}$$

where δ_{ij} is the Kronecker delta, and $w = \zeta^\vee - \zeta$ is the deflection in the direction of the normal vector $\mathbf{n}^p$.

An analysis of the relations (1.4) shows that the terms appearing in parentheses in the expression for ε_{ij} vanish with diminution of the size of the shell surface. The retention of these terms permits the approximate strain measures valid for thin

S. A. Chaplygina, Siberian Scientific–Research Institute of Aviation, 630051 Novosibirsk. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 2, pp. 150-158, March-April, 1995. Original article submitted March 31, 1994.

0021-8944/95/3602-0279$12.50

TABLE 1

Grid	Simply supported edges (k = 4)		Two simply supported and two free edges (k = 1)	
	Model A	Model B	Model A	Model B
2 × 2	3,00	3,46	0,943	1,140
4 × 4	3,55	3,88	0,985	1,035
8 × 8	3,88	3,92	—	—

plates to be accurately extended to strain relations for shallow (small-curvature) shells [3]. The employment of more complicated deformation models for the development of numerical algorithms would be inexpedient both in connection with its laborious numerical implementation and from the viewpoint of the main concepts of the finite element method.

2. We consider a three-node finite element of triangular shape with five degrees of freedom at each node: three displacements and two angles of rotation of the normal vector. For unknown functions characterizing the geometry of the shell element before and after deformation we assume the following approximations:

$$\mathbf{r}^p = L_k \mathbf{r}_k, \quad \mathbf{r}^{p\vee} = L_k \mathbf{r}_k^{\vee}, \quad \zeta = -N^{ik}\vartheta_{ik}, \quad \zeta^{\vee} = -N^{ik}\vartheta_{ik}^{\vee}$$
$$(i = 1,2, \qquad k = 1,2,3). \tag{2.1}$$

Here $\mathbf{r}_k$ are the nodal values of the radius vector, ϑ_{ik} is the cosine of the angle between the normal vector at the k-th node and the coordinate vector $\mathbf{r}_{,i}$, and L_k and N^{ik} are the area coordinates and cubical shape functions having the form [1]

$$L_k = b_{ki}\xi_i + b_{k3},$$
$$b_{i1} = \frac{\xi_{2j} - \xi_{2k}}{2F}, \qquad b_{i2} = \frac{\xi_{1k} - \xi_{1j}}{2F}, \qquad b_{i3} = \frac{\xi_{1j}\xi_{2k} - \xi_{1k}\xi_{2j}}{2F},$$
$$N^{1i} = 2F(b_{k2}\eta_{1i} - b_{j2}\eta_{2i}), \qquad N^{2i} = 2F(b_{j1}\eta_{2i} - b_{k1}\eta_{1i}), \tag{2.2}$$
$$\eta_{1i} = L_i^2 L_j + c_{1i}L_iL_jL_k, \qquad \eta_{2i} = L_i^2 L_k + c_{2i}L_iL_jL_k,$$
$$c_{1i} = \frac{1}{2}\frac{3h_i^2 - h_j^2 + h_k^2}{h_i^2 + h_j^2 + h_k^2}, \qquad c_{2i} = \frac{1}{2}\frac{3h_i^2 - h_k^2 + h_j^2}{h_i^2 + h_j^2 + h_k^2},$$

where F is the area of the element, ξ_{ij} is the i-th coordinate of the j-th node of the element, and h_i is the length of the element side opposite to the i-th node. The relations (2.2) obey the rule of cyclic permutation of the ices i, j, and k.

Substituting the expressions (2.1) into (1.4) and averaging the components of the strain tensor within the limits of an element, we obtain the strain relations for a shell finite element:

$$\varepsilon_{ij} = e_{ij} + \frac{1}{2}u_{ij}^{mrst}(\vartheta_{mr}^{\vee}\vartheta_{st}^{\vee} - \vartheta_{mr}\vartheta_{st}),$$
$$e_{ij} = \frac{1}{2}(x_{kt}^{p\vee}x_{kr}^{p\vee}b_{ti}b_{rj} - \delta_{ij}),$$
$$æ_{ij} = N_{,ij}^{mr}\theta_{mr}, \quad \theta_{mr} = \vartheta_{mr}^{\vee} - \vartheta_{mr}, \quad \vartheta_{mr}^{\vee} = b_{tm}\lambda_{kr}^{n\vee}x_{kt}^{p\vee}, \tag{2.3}$$
$$u_{ij}^{mrst} = \frac{1}{2F}\int_F (N_{,i}^{mr}N_{,j}^{st} + N_{,j}^{mr}N_{,i}^{st})\,dF$$
$$(i,j,m,s = 1,2, \qquad k,r,t = 1,2,3).$$

Here x_{kr}^p is the k-th coordinate of the r-th node of the element, and θ_{mr} are the nodal values of the variations of the cosines of the angles of inclination of the normal vector (m enumerates the cosines number, and r enumerates the nodes).

Hereafter the finite-element model based on the strain relations (2.3) will be called model A. Model B implies finite-element calculations utilizing the simplified variant of the strain relations

TABLE 2

Grid	Deflection $w(0)\cdot 10^2$, m		Deflection $w(\pi/2)\cdot 10^2$, m	
	Model A	Model B	Model A	Model B
2 × 2	0,2891	0,2865	0,2752	0,2629
3 × 3	0,2990	0,2832	0,2737	0,2577
4 × 4	0,2972	0,2853	0,2675	0,2565
5 × 5	0,2953	0,2864	0,2638	0,2557
6 × 6	0,2929	0,2871	0,2605	0,2552
7 × 7	0,2921	0,2874	0,2592	0,2548

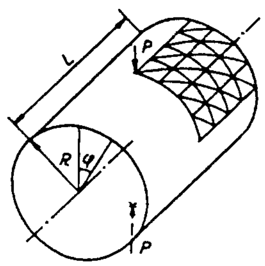

Fig. 1

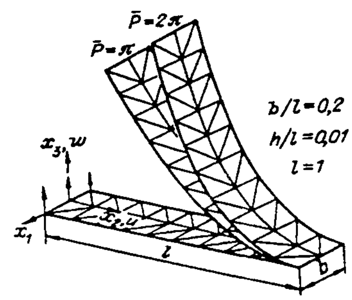

Fig. 2

$$\varepsilon_{ij} = e_{ij}, \qquad æ_{ij} = N^{mr}_{,ij}\,\theta_{mr}. \tag{2.4}$$

A comparison of two variants of the deformation models based on the relations (2.3) and (2.4) shows that refinement concerns the distribution of strains within an element and not its spatial motion. Both finite-element variants ensure a calculation of strain state of thin shells of arbitrary shape undergoing finite elastic displacements and rotations. It is of interest to study the questions of convergence and accuracy of solutions given by models A and B in classical problems of the statics of thin shells.

3. We use the energy method to solve the problem of determining an equilibrium state. We write the expression for the potential energy of a shell element:

$$\Pi = \frac{1}{2}\int_V \sigma^z_{ij}(\varepsilon^z_{ij} - \alpha_{ij}T)\,dV \tag{3.1}$$

(α_{ij} is the tensor of the coefficients of thermal expansion, σ^z_{ij} is the stress tensor, and T is the temperature change).

The thermoelasticity relations have the form

$$\sigma^z_{ij} = a_{ijkl}(z)(\varepsilon^z_{kl} - \alpha_{kl}T). \tag{3.2}$$

Substituting (3.2) into (3.1) and taking into account the Kirchhoff–Love hypotheses $\varepsilon^z_{ij} = \varepsilon_{ij} + z æ_{ij}$ and the symmetry property of the tensor of elastic constants ($a_{ijkl} = a_{klij}$), we obtain

$$\Pi = \frac{1}{2}\int_F (B_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + 2C_{ijkl}\varepsilon_{ij}æ_{kl} + D_{ijkl}æ_{ij}æ_{kl})\,dF - \\ - \int_F Q_{ij}\varepsilon_{ij}dF - \int_F R_{ij}æ_{ij}dF + \text{const}. \tag{3.3}$$

Here,

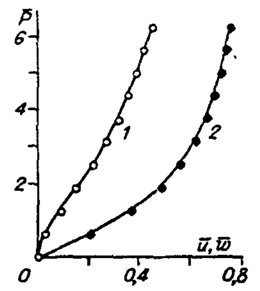

Fig. 3

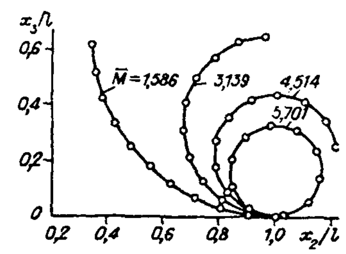

Fig. 4

$$B_{ijkl} = \int_h a_{ijkl} dz; \quad C_{ijkl} = \int_h a_{ijkl} z\, dz; \quad D_{ijkl} = \int_h a_{ijkl} z^2\, dz;$$
$$Q_{ij} = \int_h a_{ijkl}\alpha_{kl} T\, dz; \quad R_{ij} = \int_h a_{ijkl}\alpha_{kl} T z\, dz$$

(Q_{ij} and R_{ij} are the temperature forces and moments). Introducing the strain relations (2.3) into (3.3) and integrating over the area of an element, we obtain an expression for the potential energy of an element in matrix notation:

$$\Pi = \frac{1}{2}(\varepsilon^{\mathrm{T}}\mathbf{K}_\varepsilon \varepsilon + 2\varepsilon^{\mathrm{T}}\mathbf{K}_{\varepsilon\theta}\theta + \theta^{\mathrm{T}}\mathbf{K}_\theta\theta) - \varepsilon^{\mathrm{T}}\mathbf{Q}_\varepsilon - \theta^{\mathrm{T}}\mathbf{Q}_\theta + \text{const},$$
$$\varepsilon^{\mathrm{T}} = |\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}|, \quad \theta^{\mathrm{T}} = |\theta_{11}, \theta_{21}, \theta_{12}, \theta_{22}, \theta_{13}, \theta_{23}|,$$

where $\mathbf{K}_\varepsilon$, $\mathbf{K}_{\varepsilon\theta}$, and $\mathbf{K}_\theta$ are submatrices of the element stiffness matrix, and $\mathbf{Q}_\varepsilon$ and $\mathbf{Q}_\theta$ are the heat-input vectors calculated from the formulas

$$\mathbf{K}_\varepsilon = \mathbf{B}F, \quad \mathbf{K}_{\varepsilon\theta} = \mathbf{C}\int_F \mathbf{N}'' dF, \quad \mathbf{K}_\theta = \int_F (\mathbf{N}'')^{\mathrm{T}}\mathbf{D}\mathbf{N}'' dF,$$
$$\mathbf{B} = \int_h \mathbf{a}\, dz, \quad \mathbf{C} = \int_h \mathbf{a} z\, dz, \quad \mathbf{D} = \int_h \mathbf{a} z^2\, dz, \quad \mathbf{Q}_\varepsilon = F\int_h \mathbf{a}\alpha T\, dz, \tag{3.4}$$
$$\mathbf{Q}_\theta = \int_F (\mathbf{N}'')^{\mathrm{T}} dF \int_h \mathbf{a}\alpha T z\, dz, \quad \alpha^{\mathrm{T}} = |\alpha_{11}, \alpha_{22}, \alpha_{12}|,$$

$$\mathbf{a} = \begin{bmatrix} a_{1111} & a_{1122} & 2a_{1112} \\ & a_{2222} & 2a_{2212} \\ \text{sym} & & 4a_{1212} \end{bmatrix},$$

$$\mathbf{N}'' = \begin{bmatrix} N^{11}_{,11} & N^{21}_{,11} & N^{12}_{,11} & N^{22}_{,11} & N^{13}_{,11} & N^{23}_{,11} \\ N^{11}_{,22} & N^{21}_{,22} & N^{12}_{,22} & N^{22}_{,22} & N^{13}_{,22} & N^{23}_{,22} \\ N^{11}_{,12} & N^{21}_{,12} & N^{12}_{,12} & N^{22}_{,12} & N^{13}_{,12} & N^{23}_{,12} \end{bmatrix}.$$

Here, T is the average temperature change in the element, $\mathbf{a}$ is the matrix of elastic constants, and α is the vector of coefficients of temperature expansion. Integration over the area of an element is carried out numerically by the Gaussian cubature formulas [4].

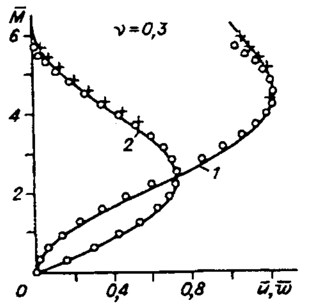

Fig. 5

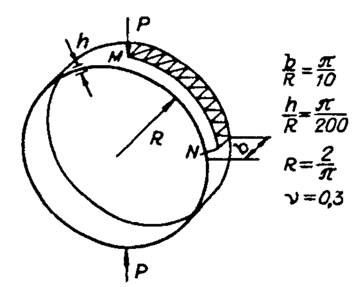

Fig. 6

In the case of an isotropic shell the components of the matrix **a** have the form

$$a_{1111} = a_{2222} = E/(1 - \nu^2), \qquad a_{1122} = \nu a_{1111},$$
$$a_{1112} = a_{2212} = 0, \qquad a_{1212} = 0{,}5E/(1 + \nu).$$

4. To formulate the equations of equilibrium, stability, and the iterative solution process, the coefficients of the first and second variations of the potential energy of a nonlinear discrete system need to be calculated.

To obtain a computational algorithm, we introduce two levels of the variable discrete parameters represented by the vector of generalized elastic displacements $\mathbf{u}^*$ and the vector of generalized coordinates $\mathbf{u}^{**} = \mathbf{q}$:

$$(\mathbf{u}^*)^{\mathsf{T}} = |\mathbf{e}^{\mathsf{T}}, \boldsymbol{\theta}^{\mathsf{T}}|, \quad \mathbf{e}^{\mathsf{T}} = |e_{11}, e_{22}, e_{12}|,$$
$$(\mathbf{u}^{**})^{\mathsf{T}} = \mathbf{q}^{\mathsf{T}} = |\mathbf{q}_1^{\mathsf{T}}, \mathbf{q}_2^{\mathsf{T}}, \mathbf{q}_3^{\mathsf{T}}|, \quad \mathbf{q}_i^{\mathsf{T}} = |x^{\vee}_{1i}, x^{\vee}_{2i}, x^{\vee}_{3i}, \varphi^{\vee}_{1i}, \varphi^{\vee}_{2i}|$$

($\varphi^{\vee}_{ij}$ are the angles of rotation of the normal vector $\mathbf{n}^{\vee}_j$).

The solving algorithm for the nonlinear problem is based on the stationary conditions for the total potential energy of the discrete system:

$$\mathbf{H}\delta\mathbf{q} + \mathbf{g} - \mathbf{Q} = 0 \tag{4.1}$$

(**H** and **g** are the Hess matrix and the gradient of the potential energy of the ensemble of finite elements, **Q** is the vector of generalized external forces).

The Hess matrix $\mathbf{h}^{**}$ and the gradient $\mathbf{g}^{**}$ of a finite element are calculated by the formulas

$$\mathbf{g}^{**} = (\mathbf{u}^{*\prime})^{\mathsf{T}}\mathbf{g}^*, \quad \mathbf{h}^{**} = (\mathbf{u}^{*\prime})^{\mathsf{T}}\mathbf{h}^*\mathbf{u}^{*\prime} + g_i^*\mathbf{u}_i^{*\prime\prime},$$

where $\mathbf{g}^*$ and $\mathbf{h}^*$ are the gradient and the Hess matrix of the first level, and $\mathbf{u}^{*\prime}$ and $\mathbf{u}^{*\prime\prime}$ are the matrices of the first and second derivatives of the components of the vector $\mathbf{u}^*$ with respect to the components of the vector $\mathbf{u}^{**}$ as defined by the relations

$$(\mathbf{g}^*)^{\mathsf{T}} = |\mathbf{g}_e^{\mathsf{T}}, \mathbf{g}_\theta^{\mathsf{T}}| = |g_1^*, \ldots, g_9^*|, \quad \mathbf{h}^* = \begin{bmatrix} \mathbf{h}_e & \mathbf{h}_{e\theta} \\ \mathbf{h}_{e\theta}^{\mathsf{T}} & \mathbf{h}_\theta \end{bmatrix},$$
$$\mathbf{g}_e = \mathbf{K}_\epsilon\boldsymbol{\epsilon} + \mathbf{K}_{\epsilon\theta}\boldsymbol{\theta}, \quad \mathbf{g}_\theta = \mathbf{K}_{\epsilon\theta}^{\mathsf{T}}\boldsymbol{\epsilon} + \mathbf{K}_\theta\boldsymbol{\theta} + \mathbf{A}\mathbf{g}_e,$$
$$\mathbf{h}_e = \mathbf{K}_\epsilon, \quad \mathbf{h}_{e\theta} = \mathbf{K}_{\epsilon\theta} + \mathbf{K}_\epsilon\mathbf{A}^{\mathsf{T}},$$
$$\mathbf{h}_\theta = \mathbf{K}_\theta + \mathbf{A}\mathbf{K}_{\epsilon\theta} + \mathbf{K}_{\epsilon\theta}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} + \mathbf{A}\mathbf{K}_\epsilon\mathbf{A}^{\mathsf{T}} + g_1^*\mathbf{u}_{11} + g_2^*\mathbf{u}_{22} + g_3^*\mathbf{u}_{12}, \tag{4.2}$$
$$\mathbf{A} = |\mathbf{u}_{11}\boldsymbol{\vartheta}^{\vee}, \mathbf{u}_{22}\boldsymbol{\vartheta}^{\vee}, \mathbf{u}_{12}\boldsymbol{\vartheta}^{\vee}|, \quad \mathbf{u}_{ij} = \frac{1}{2F}\int\limits_F (\mathbf{N}_{,i}\mathbf{N}_{,j}^{\mathsf{T}} + \mathbf{N}_{,j}\mathbf{N}_{,i}^{\mathsf{T}})\, dF,$$
$$\mathbf{N}^{\mathsf{T}} = |N^{11}, N^{21}, N^{12}, N^{22}, N^{13}, N^{23}|, \quad \boldsymbol{\vartheta}^{\vee\mathsf{T}} = |\vartheta^{\vee}_{11}, \vartheta^{\vee}_{21}, \vartheta^{\vee}_{12}, \vartheta^{\vee}_{22}, \vartheta^{\vee}_{13}, \vartheta^{\vee}_{23}|.$$

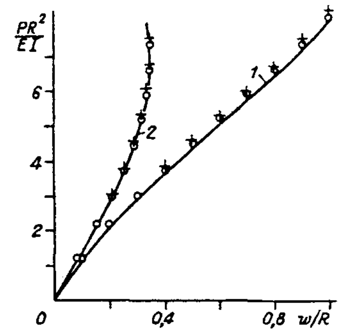

Fig. 7

The integration for $\mathbf{u}_{ij}$ in (4.2) is carried out numerically by the Gaussian method [4]. The nonzero components of the matrix $\mathbf{u}^{*\prime}$ have the form

$$\begin{aligned}
&\frac{\partial e_{ij}}{\partial x^{\vee}_{km}} = \frac{1}{2} x^{\vee}_{ks}(b_{sj}b_{mi} + b_{si}b_{mj}),\\
&\frac{\partial \vartheta^{\vee}_{rs}}{\partial \varphi^{\vee}_{is}} = x^{\vee}_{km} b_{mr} \lambda^{\vee}_{iks},\\
&\frac{\partial \vartheta^{\vee}_{rs}}{\partial x^{\vee}_{km}} = b_{mr} \lambda^{n\vee}_{ks} \quad (i, j, r = 1, 2, \;\; k, m, s = 1, 2, 3)
\end{aligned} \tag{4.3}$$

($\lambda^{\vee}_{iks}$ are the direction cosines of two auxiliary unit vectors, which together with the normal vector $\mathbf{n}^{\vee}_{s}$ form a right-handed triad).

The nonzero components of the matrices of second derivatives $\mathbf{u}^{*\prime\prime}_{i}$ are calculated from the formulas

$$\begin{aligned}
&\frac{\partial^2 e_{ij}}{\partial x^{\vee}_{km} \partial x^{\vee}_{kl}} = \frac{1}{2}(b_{lj}b_{mi} + b_{li}b_{mj}),\\
&\frac{\partial^2 \vartheta^{\vee}_{rs}}{\partial \varphi^{\vee 2}_{is}} = -x^{\vee}_{km} b_{mr} \lambda^{n\vee}_{ks},\\
&\frac{\partial^2 \vartheta^{\vee}_{rs}}{\partial \varphi^{\vee}_{is} \partial x^{\vee}_{km}} = b_{mr} \lambda^{\vee}_{iks} \quad (i, j, l, r = 1, 2, \;\; k, m, s = 1, 2, 3).
\end{aligned} \tag{4.4}$$

The relations (4.3) and (4.4) are used for cyclic calculation of the gradient and the Hess matrix of the potential energy of a shell finite element, making it possible to obtain a compact computational algorithm.

In the process of the iterative solution of the equilibrium equations according to the scheme (4.1) it is necessary to calculate new values of the unknown quantities after each iteration. In the case of finite displacements and rotations the new values of the nodal unknowns are calculated from the formulas

$$\mathbf{r}^{\vee\vee} = \mathbf{r}^{\vee} + \delta \mathbf{r}^{\vee}, \quad \mathbf{n}^{\vee\vee} = \mathbf{n}^{\vee} \cos \delta n^{\vee} + \delta \mathbf{n}^{\vee} \sin(\delta n^{\vee})/\delta n^{\vee},$$

where $\delta n^{\vee}$ is the magnitude of the vector $\delta \mathbf{n}^{\vee}$.

The algorithm formulated above enables one to determine the equilibrium states and to investigate their stability in accordance with the criterion of positive definiteness of the second energy variation.

5. We now consider the stability problem for a simply supported square plate with the side b compressed uniaxially by forces p. We assume that $v = 0$. With symmetry conditions taken into account we obtain analytical expressions for the stability equations for one quarter of the plate divided into two triangular elements

$$\mathbf{K}\mathbf{q} - pb^2D^{-1}(\mathbf{B}+\mathbf{C})\mathbf{q} = 0, \quad \mathbf{q}^T = |2w/b, \varphi|, \tag{5.1}$$

where w is the bending deflection at the center of the plate, φ is the angle of rotation of the normal vector and D is the flexural stiffness. The nonzero components of the matrices in (5.1) have the form

$$K_{11} = 84, \ K_{12} = K_{21} = 38, \ K_{22} = 29, \ B_{11} = 1,$$
$$C_{11} = 3/20, \ C_{12} = C_{21} = 1/15, \ C_{22} = 73/720.$$

The matrix C reflects the influence of the additional terms in the strain relations (1.4) characterizing model A. From the condition $\det(\mathbf{K} - pb^2D^{-1}(\mathbf{B}+\mathbf{C})) = 0$ we obtain the critical load parameter $k = p_cb^2/\pi^2D$, the exact value of which is $k_e = 4$ [5]:

$$k = \frac{192}{323\pi^2}\left(276 - \sqrt{\frac{102287}{2}}\right) = 3{,}00 \qquad \text{(model A)},$$

$$k = \frac{992}{29\pi^2} = 3{,}46 \qquad \text{(model B)}$$

(p_c is the critical load).

Setting $\varphi = 0$ in (5.1), one can also find a solution of the problem for a plate with clamped edges. In this case the following values for the critical load parameter are obtained ($k_e = 10.07$ [5]).

$$k = \frac{1680}{23\pi^2} = 7{,}40 \qquad \text{(model A)},$$

$$k = \frac{84}{\pi^2} = 8{,}51 \qquad \text{(model B)}.$$

The results indicate that the error corresponding to a coarse discretization grid is equal to 25% and 15% for models A and B, respectively.

Table 1 shows the results of a computer analysis of the convergence of the finite-element solutions carried out on a computer. The analysis of the results demonstrates the advantages of model A in the case of two simply supported and two free edges. For example, for a 2 × 2 grid the error of the critical parameter is 5.7% and 14% for models A and B, respectively.

The linear solution has been analyzed for a cylindrical shell with free edges loaded with two point forces P = 453.6 N characterized by the parameters L = 0.2629 m, R = 0.1258 m, $h = 0.2387 \cdot 10^{-2}$ m, $E = 0.738 \cdot 10^5$ MPa, and v = 0.3125 (Fig. 1). Owing to symmetry properties, discretization for one-eighth of the shell is sufficient. The results of the investigation of convergence in determining the deflections are listed in Table 2. For comparison we give values obtained in different papers for the deflection w(0) at the point of application of the force: $0.276 \cdot 10^{-2}$ m [6], $0.287 \cdot 10^{-2}$ m [7], $0.279 \cdot 10^{-2}$ m [8], and $2886 \cdot 10^{-2}$ m [9]. An analysis of the results of the calculations shows that both models converge, model A giving an upper bound of the solution, and model B a lower bound.

The nonlinear algorithm has been used to solve the problem of bending of a thin cantilevered strip loaded by a transverse force P and bending moment M. The parameters of the plate and the finite-element grid are given in Fig. 2. The equilibrium states of the strip for $\bar{P} = \pi$ and $\bar{P} = 2\pi$ ($\bar{P} = Pl^2/EI$, $I = bh^3/12$) are also presented here.

Figure 3 shows nonlinear characteristic curves relating the load parameter $\bar{P}$ to the displacement $\bar{u} = u/l$ (curve 1) and deflection $\bar{w} = w/l$ (curve 2) of the free end (the curves represent solutions obtained by a beam scheme [10], and the light and dark circles represent the finite-element solutions for models A and B, respectively). Both models give the results that coincide practically with the beam solutions.

The equilibrium configurations of the strip for various values of the bending moment parameter $\bar{M} = Ml/EI$ are presented in Fig. 4. Figure 5 shows characteristic curves of the displacement $\bar{u}$ and deflection $\bar{w}$ (curves 1 and 2) corresponding to the solution. Here, in accordance with the notation adopted above, we show the values calculated by means of the finite-element models under consideration. For small values of the load parameter ($\bar{M} < 1$) all three solutions are close to one another. When the load increases ($\bar{M} > 1$), the shell models become stiffer than a beam.

The nonlinear deformation of shells has been investigated by the above-described method expounded above in the problem of bending of a narrow ring loaded by two diametrically directed point forces P. The grid and geometrical parameters of the shell are given in Fig. 6.

In Fig. 7 curves 1 and 2 represent the nonlinear characteristics of the deflections of points M and N obtained on the basis of the beam scheme. The dots represent the solution corresponding to the finite-element model A, and the crosses correspond to model B. In this problem better agreement with the beam solution (solid curves) is observed for model A.

An analysis of the results of the calculations shows that the development of an optimal finite-element model depends first of all on the extent to which the nonlinear external connections of the finite elements are reflected, and to which standard convergence criteria [2] are satisfied. This question is intimately related to the choice of generalized elastic displacements in nonlinear problems reflecting the three-dimensional character of actual geometrical relations existing in the shell, and also to the description of large displacements of such a geometrical object. The more accurate approximation of the strain distribution within an element is not as decisive to the achievement of a more accurate solution as the diminution of element size. It should be noted that the use of the finite-element model A in calculations leads to large expenditures of time. For example, the time required to solve the problem of bending of a cantilevered strip subjected to a force $\bar{P}$ varying from $\bar{P} = 0$ to $\bar{P} = 2\pi$ in 10 loading steps using model A is about twice the time required for model B.

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